A Generalized Soliton Solution of the Konopelchenko-Dubrovsky Equation using He's Exp-Function Method

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In this paper, J. H. He's exp-function method is used to obtain a generalized soliton solution with some free parameters of the Konopelchenko-Dubrovsky equation. Suitable choice of parameters in the generalized solution leads to Wazwaz's solution [Mathematical and Computer Modelling **45**, 473 (2007)]. The result shows that He's method is very effective and convenient.

Key words: Soliton; Exp-Function Method; Konopelchenko-Dubrovsky Equation.

1. Introduction

The growing recognition that the way to solve exact soliton solutions of nonlinear equations is a crucial factor in the progress of nonlinear dynamics and a key access to understand nonlinear equations to a large extent and has fuelled much research on the determination of soliton solutions. In particular, over the past two decades many studies have focused on some special methods for some kinds of nonlinear equations with advantages on the one hand and disadvantages on the other hand, such as the variational iteration method [1 – 8], the homotopy perturbation method [9 – 16], the tanh-method [17], the F-expansion method [18 – 22]. A complete review is available in [23, 24].

Recently a novel approach called exp-function method [25 – 33] emerged not only as a new mathematical tool for searching for solitary and periodic solutions, but also as a unifying frame in which other said methods could be presented as complimentary view points. The basic idea of the exp-function method first appeared in He's monograph [25], and was systematically illustrated in [26 – 29].

In this paper we will show the effectiveness and convenience using the method by an illustrating example.

2. The Konopelchenko-Dubrovsky Equation

Consider the Konopelchenko-Dubrovsky (KD) equation [34 – 36]

$$u_t - u_{xxx} - 6buu_x + \frac{3}{2}a^2u^2u_x - 3v_y + 3au_xv = 0, \quad (1)$$

$$u_y = v_x, \quad (2)$$

where a and b are real parameters. The system has been studied widely via various methods [34 – 36].

Define η as

$$\eta = kx + ly + \omega t. \quad (3)$$

So (1) and (2) turn out to have the following forms:

$$\begin{aligned} &\omega u' - k^3u''' - 6bkuu' + \frac{3}{2}a^2ku^2u' \\ &- 3/v' + 3akvu' = 0, \end{aligned} \quad (4)$$

$$lu' = kv', \quad (5)$$

where the prime denotes the differential with respect to η.

Integrating (5) with respect to η, we find

$$lu = kv. \quad (6)$$

Substituting (5) and (6) into (4), we obtain

$$\begin{aligned} &\left(\omega - 3\frac{l^2}{k}\right)u' + (3al - 6bk)uu' \\ &+ \frac{3}{2}a^2ku^2u' - k^3u''' = 0. \end{aligned} \quad (7)$$

According to He's exp-function method [25 – 29], we assume that the solution can be expressed in the following form:

$$u(\eta) = \frac{a_c\exp(c\eta) + \cdots + a_{-d}\exp(-d\eta)}{a_p\exp(p\eta) + \cdots + a_{-q}\exp(-q\eta)}. \quad (8)$$

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To determine values of c and p, we balance the linear term of highest order in (6) with the highest order nonlinear term. By simple calculation, we have

$$u''' = \frac{c_1 \exp[(c+3p)\eta] + \cdots}{c_2 \exp[(4p)\eta] + \cdots} \tag{9}$$

and

$$u^2 u' = \frac{c_3 \exp[(3c+p)\eta] + \cdots}{c_4 \exp[(4p)\eta] + \cdots}, \tag{10}$$

where c_i are determined coefficients only for simplicity.

Balancing the highest order of the exp-function in (9) and (10), we have

$$3c + p = c + 3p, \tag{11}$$

which leads to the result

$$p = c. \tag{12}$$

The values of d and q can be determined in a similar way. We balance the linear term of lowest order in (7),

$$u''' = \frac{\cdots + d_1 \exp[(-d-3q)\eta]}{\cdots + d_2 \exp[(-4q)\eta]}, \tag{13}$$

with the nonlinear term

$$u^2 u' = \frac{\cdots + d_3 \exp[(-3d+q)\eta]}{\cdots + d_4 \exp[(-4q)\eta]}, \tag{14}$$

where d_i are determined coefficients only for simplicity.

Balancing the lowest order of the exp-function in (13) and (14), we have

$$-d - 3q = -3d - q, \tag{15}$$

which leads to the result

$$d = q. \tag{16}$$

For simplicity, we set $p = c = 1$ and $q = d = 1$; so (8) reduces to

$$u(\eta) = \frac{a_1 e^{\eta} + a_0 + a_{-1} e^{-\eta}}{b_1 e^{\eta} + b_0 + b_{-1} e^{-\eta}}. \tag{17}$$

Substituting (17) into (7), and by the help of Maple, we have

$$\begin{aligned} &\frac{1}{A}(c_3 e^{3\eta} + c_2 e^{2\eta} + c_1 e^{1\eta} + c_0 \\ &+ c_{-1} e^{-\eta} + c_{-2} e^{-2\eta} + c_{-3} e^{-3\eta}) = 0, \end{aligned} \tag{18}$$

where

$$A = 2k(b_1 e^{\eta} + b_0 + b_{-1} e^{-\eta})^4,$$

$$\begin{aligned} c_3 = &-2\omega k a_0 b_1^3 - 6l^2 a_1 b_0 b_1^2 + 2k^4 a_0 b_1^3 \\ &+ 6kala_1^2 b_0 b_1 + 12bk^2 a_1 a_0 b_1^2 - 12bk^2 a_1^2 b_0 b_1 \\ &+ 3a^2 k^2 a_1^3 b_0 - 6kala_1 a_0 b_1^2 - 2k^4 a_1 b_1^2 b_0 \\ &- 3a^2 k^2 a_1^2 a_0 b_1 + 2\omega k a_1 b_0 b_1^2 + 6l^2 a_0 b_1^3, \end{aligned}$$

$$\begin{aligned} c_2 = {} & 16k^4 a_{-1} b_1^3 - 4\omega k a_0 {b_1}^2 b_0 - 12kala_1 a_{-1} {b_1}^2 \\ & -12l^2 a_1 b_{-1} {b_1}^2 + 6kala_1{}^2 b_0 2 - 4\omega k a_{-1} {b_1}^3 \\ & +6a^2k^2 {a_1}^2 a_0 b_0 + 24bk^2 a_1 a_{-1} {b_1}^2 + 4\omega k a_1 b_{-1} {b_1}^2 \\ & -6a^2k^2 a_1 {a_0}^2 b_1 + 8k^4 a_1 b_1 {b_0}^2 + 12bk^2 {a_0}^2 {b_1}^2 \\ & +12l^2 a_{-1} {b_1}^3 - 24bk^2 {a_1}^2 b_{-1} b_1 + 12kala_1{}^2 b_{-1} b_1 \\ & -12l^2 a_1 {b_0}^2 b_1 - 8k^4 a_0 {b_1}^2 b_0 - 16k^4 a_1 {b_1}^2 b_{-1} \\ & -6kala_0{}^2 {b_1}^2 + 12l^2 a_0 {b_1}^2 b_0 + 4\omega k a_1 {b_0}^2 b_1 \\ & +6a^2k^2 {a_1}^3 b_{-1} - 12bk^2 {a_1}^2 {b_0}^2 - 6a^2k^2 {a_1}^2 a_{-1} b_1, \end{aligned}$$

$$\begin{aligned} c_1 = {} & 15a^2k^2 {a_1}^2 a_0 b_{-1} - 6l^2 a_1 {b_0}^3 - 2k^4 a_1 {b_0}^3 \\ & +36k^4 a_1 b_1 b_0 b_{-1} + 6l^2 a_0 b_1 {b_0}^2 + 2\omega k a_1 {b_0}^3 \\ & -46k^4 a_0 {b_1}^2 b_{-1} + 12\omega k a_1 b_0 b_1 b_{-1} - 2\omega k a_0 b_1 {b_0}^2 \\ & -10\omega k a_{-1} {b_1}^2 b_0 + 30l^2 a_{-1} {b_1}^2 b_0 + 3a^2k^2 {a_0}^2 a_1 b_0 \\ & -12bk^2 a_0 a_1 {b_0}^2 - 36l^2 a_1 b_0 b_1 b_{-1} + 12kala_1 a_0 b_1 b_{-1} \\ & -18kala_0 a_{-1} {b_1}^2 + 10k^4 a_{-1} {b_1}^2 b_0 \\ & -24bk^2 a_1 a_0 b_1 b_{-1} + 18kala_1{}^2 b_0 b_{-1} + 2k^4 a_0 b_1 {b_0}^2 \\ & +6kala_0 a_1 {b_0}^2 - 3a^2k^2 {a_0}^3 b_1 - 36bk^2 {a_1}^2 b_0 b_{-1} \\ & -6kala_0{}^2 b_1 b_0 + 3a^2k^2 {a_1}^2 a_{-1} b_0 + 6l^2 a_0 {b_1}^2 b_{-1} \\ & -18a^2k^2 a_1 a_0 a_{-1} b_1 - 2\omega k a_0 {b_1}^2 b_{-1} \\ & +36bk^2 a_0 a_{-1} {b_1}^2 + 12bk^2 {a_0}^2 b_1 b_0 \\ & +24bk^2 a_1 a_{-1} b_1 b_0 - 12kala_1 a_{-1} b_1 b_0, \end{aligned}$$

$$\begin{aligned} c_0 = {} & -48bk^2 a_1 a_0 b_{-1} b_0 + 64k^4 a_1 b_1 {b_{-1}}^2 \\ & +24l^2 a_{-1} {b_1}^2 b_{-1} - 24l^2 a_1 b_{-1} {b_0}^2 b_1 + 24l^2 a_{-1} b_1 {b_0}^2 \\ & -8k^4 a_1 {b_0}^2 b_{-1} + 12kala_1{}^2 {b_{-1}}^2 - 24bk^2 {a_1}^2 {b_{-1}}^2 \\ & +24kala_1 a_0 b_{-1} b_0 - 12kala_{-1}{}^2 {b_1}^2 - 24l^2 a_1 {b_0}^2 b_{-1} \\ & +12a^2k^2 {a_1}^2 a_{-1} b_{-1} + 12a^2k^2 a_1 {a_0}^2 b_{-1} \\ & -24kala_0 a_{-1} b_1 b_0 + 48bk^2 a_0 a_{-1} b_1 b_0 \\ & -12a^2k^2 a_1 {a_{-1}}^2 b_1 + 8k^4 a_{-1} b_1 {b_0}^2 \\ & -12a^2k^2 {a_0}^2 a_{-1} b_1 - 64k^4 a_{-1} {b_1}^2 b_{-1} \\ & +8\omega k a_1 {b_0}^2 b_{-1} + 8\omega k a_1 {b_{-1}}^2 b_1 \\ & -8\omega k a_{-1} {b_1}^2 b_{-1} - 8\omega k a_{-1} b_1 {b_0}^2, \end{aligned}$$

$$c_{-1} = 36l^2a_{-1}b_1b_0b_{-1} + 46k^4a_0b_1{b_{-1}}^2 -10k^4a_1b_0{b_{-1}}^2 + 6l^2a_{-1}{b_0}^3 - 30l^2a_1b_0{b_{-1}}^2 +12bk^2a_0a_{-1}{b_0}^2 - 2k^4a_0b_{-1}{b_0}^2 - 36k^4a_{-1}b_1b_0b_{-1} -2\omega ka_{-1}b_03 + 24bk^2a_0a_{-1}b_1b_{-1} + 3a^2k^2{a_0}^3b_{-1} -15a^2k^2a_0{a_{-1}}^2b_1 + 10\omega ka_1b_0{b_{-1}}^2 - 6l^2a_0b_{-1}{b_0}^2 +2\omega ka_0b_1{b_{-1}}^2 + 36bk^2{a_{-1}}^2b_1b_0 - 6l^2a_0b_1{b_{-1}}^2 -12\omega ka_{-1}b_1b_0b_{-1} - 12bk^2{a_0}^2b_{-1}b_0 -36bk^2a_1a_0{b_{-1}}^2 - 3a^2k^2a_1{a_{-1}}^2b_0 + 2k^4a_{-1}{b_0}^3 -3a^2k^2{a_0}^2a_{-1}b_0 - 24bk^2a_{-1}a_1b_{-1}b_0 +18kala_1a_0{b_{-1}}^2 + 12kala_{-1}a_1b_{-1}b_0 +2\omega ka_0b_{-1}{b_0}^2 + 6kal{a_0}^2b_{-1}b_0 - 6kala_0a_{-1}{b_0}^2 +18a^2k^2a_1a_{-1}a_0b_{-1} - 12kala_0a_{-1}b_1b_{-1} -18kal{a_{-1}}^2b_1b_0,$$

$$c_{-2} = -6kal{a_{-1}}^2{b_0}^2 + 24bk^2{a_{-1}}^2b_1b_{-1} +6kal{a_0}^2{b_{-1}}^2 - 4\omega ka_{-1}b_1{b_{-1}}^2 - 6a^2k^2a_0{a_{-1}}^2b_0 +12l^2a_{-1}b_1{b_{-1}}^2 - 24bk^2a_{-1}a_1{b_{-1}}^2 +6a^2k^2{a_0}^2a_{-1}b_{-1} + 4\omega ka_0{b_{-1}}^2b_0 +12l^2a_{-1}{b_0}^2b_{-1} + 4\omega ka_1{b_{-1}}^3 + 12kala_{-1}a_1{b_{-1}}^2 +12bk^2{a_{-1}}^2{b_0}^2 - 12bk^2{a_0}^2{b_{-1}}^2 + 8k^4a_0{b_{-1}}^2b_0 +6a^2k^2{a_{-1}}^2a_1b_{-1} - 12l^2a_1{b_{-1}}^3 - 8k^4a_{-1}{b_0}^2b_{-1} -16k^4a_1{b_{-1}}^3 + 16k^4a_{-1}b_1{b_{-1}}^2 - 12l^2a_0{b_{-1}}^2b_0 -6a^2k^2{a_{-1}}^3b_1 - 12kal{a_{-1}}^2b_1b_{-1} - 4\omega ka_{-1}{b_0}^2b_{-1},$$

$$c_{-3} = -12bk^2a_{-1}a_0{b_{-1}}^2 + 12bk^2{a_{-1}}^2b_0b_{-1} -2\omega ka_{-1}b_0{b_{-1}}^2 + 2k^4a_{-1}b_0{b_{-1}}^2 + 6kala_{-1}a_0{b_{-1}}^2 +6l^2a_{-1}b_0{b_{-1}}^2 + 3a^2k^2{a_{-1}}^2a_0b_{-1} - 6l^2a_0{b_{-1}}^3 -2k^4a_0{b_{-1}}^3 - 6kal{a_{-1}}^2b_0b_{-1} - 3a^2k^2{a_{-1}}^3b_0 +2\omega ka_0{b_{-1}}^3.$$

Equating the coefficients of $\mathrm{e}^{n\eta}$ to be zero, we have

$$\begin{cases} c_3 = 0, \quad c_3 = 0, \quad c_3 = 0, \\ c_0 = 0, \\ c_{-1} = 0, \quad c_{-2} = 0, \quad c_{-3} = 0. \end{cases} \tag{19}$$

Solving the system (19) simultaneously, we get

Case 1:

$$a_1 = -\frac{b_1(-k^2a + al - 2bk)}{ka^2}, \quad b_1 = b_1,$$

$$a_{-1} = \frac{1}{4}\frac{1}{b_1k^5a^4}\Big(8b^2k^3a_0b_0a^2 - 8ba_0b_0lk^2a^3 +k^2a^3l^2b_0^2 + 4k^4ab^2b_0^2 + 4k^3a^2blb_0^2 + 2k^3a^4la_0b_0 -4k^4a^3ba_0b_0 + 12alb^2b_0^2k^2 - 6a^2l^2bb_0^2k - 2bk^3a_0^2a^4 +a^5la_0^2k^2 + 2b_0^2bk^5a^2 - b_0^2lk^4a^3 - b_0^2k^6a^3 +k^4a^5a_0^2 + a^3l^3b_0^2 - 8b^3k^3b_0^2 + 2l^2a_0b_0ka^4\Big),$$

$$b_{-1} = -\frac{1}{4}\frac{1}{b_1k^4a^2}\Big(a_0^2k^2a^4 + l^2b_0^2a^2 + 4b^2b_0^2k^2 -b_0^2k^4a^2 - 4blb_0^2ka + 2la_0b_0ka^3 - 4ba_0b_0k^2a^2\Big),$$

$$\omega = \frac{1}{2}\frac{-k^4a^2 + 9l^2a^2 - 12bkal + 12b^2k^2}{ka^2},$$

or

Case 2:

$$a_1 = -\frac{b_1(k^2a + al - 2bk)}{ka^2}, \quad b_1 = b_1,$$

$$a_{-1} = \frac{1}{4}\frac{1}{b_1k^5a^4}\Big(-k^4a^5a_0^2 - k^2a^3l^2b_0^2 -4k^4ab^2b_0^2 + b_0^2k^6a^3 + 4k^3a^2blb_0^2 - 2k^3a^4la_0b_0 +4k^4a^3ba_0b_0 + a^5la_0^2k^2 + a^3l^3b_0^2 + 12alb^2b_0^2k^2 -b_0^2lk^4a^3 - 6a^2l^2bb_0^2k + 2l^2a_0b_0ka^4 - 8ba_0b_0lk^2a^3 -2bk^3a_0^2a^4 - 8b^3k^3b_0^2 + 2b_0^2bk^5a^2 + 8b^2k^3a_0b_0a^2\Big),$$

$$b_{-1} = -\frac{1}{4}\frac{1}{b_1k^4a^2}\Big(a_0^2k^2a^4 + l^2b_0^2a^2 + 4b^2b_0^2k^2 -b_0^2k^4a^2 - 4blb_0^2ka + 2la_0b_0ka^3 - 4ba_0b_0k^2a^2\Big),$$

$$\omega = \frac{1}{2}\frac{-k^4a^2 + 9l^2a^2 - 12bkal + 12b^2k^2}{ka^2}.$$

We, therefore, obtain the generalized soliton solution

$$u(\eta) = \frac{a_1\mathrm{e}^{\eta} + a_0 + a_{-1}\mathrm{e}^{-\eta}}{b_1\mathrm{e}^{\eta} + b_0 + b_{-1}\mathrm{e}^{-\eta}}, \tag{20}$$

where the parameters are determined in Case 1 or Case 2.

If we set $k = l = \frac{2b-a}{a}$ and $a_0 = -\frac{2(-2b+a)b_0}{a^2}$ in Case 1, (20) reduces to

$$u(x,y,t) = \frac{2(2b-a)}{a^2\left(1 + \frac{b_1}{b_0}\mathrm{e}^{\frac{(2b-a)}{a}\left(x+y-\frac{4(ba-a^2-b^2)}{a^2}t\right)}\right)}. \tag{21}$$

If $k = l = -\frac{2b-a}{a}$ and $a_0 = -\frac{2(-2b+a)b_0}{a^2}$ in Case 2, our solution becomes

$$u(x,y,t) = \frac{2(2b-a)}{a^2\left(1+\frac{b_1}{b_0}\mathrm{e}^{-\frac{(2b-a)}{a}\left(x+y-\frac{4(ba-a^2-b^2)}{a^2}t\right)}\right)}. \tag{22}$$

Equations (21) and (22) are the same as Wazwaz's solution [36].

3. Conclusion

We obtained a generalized soliton solution of the Konopelchenko-Dubrovsky equation; Wazwaz's solution [36] can be considered as a special case of our result. The results show that He's exp-function method is a powerful mathematical tool for solving generalized soliton solutions.

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